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Best Approximations from Hilbert Submanifolds

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The problems of existence, uniqueness, and differentiability of best approximations from Hilbert submanifolds are considered.

1. INTRODUCTION

In this work we will consider the problems of existence, uniqueness, and differentiability of best approximations from a Hilbert submanifold, i.e., a possibly ∞-dimensional immersed submanifold M of a separable Hilbert space H. M will be given the metric induced by the immersion. For a recent survey of best approximations in Hilbert space see the short review by Berens [9].)

The main results of this paper are strengthenings and generalizations of results which are known in the finite dimensional case, see e.g., the papers by Abatzoglou [4–5]. The ∞-dimensional case requires somewhat different techniques.

There has appeared a number of results relating the metric curvature of general closed subsets of normed linear spaces to the properties of their best approximation operator. This makes it relevant to comment on the assumptions which are made in this paper.

The smoothness inherent in the concept of a submanifold makes it possible to define the metric curvature in terms of an analytic quantity, the normal curvature. This enables us to make sharp estimates of the reach of submanifolds.

It seems to be an open question exactly what degree of differentiability is needed for these results to hold; certainly the condition of C^2-differentiability used here is too strong and should probably be replaced by C^1 plus a Lipschitz condition related to metric curvature.

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2. Preliminaries

For background in differential geometry the reader is referred to [14–16]. In the following, M will be an immersed submanifold of H and will be considered as a subset of H, with the geometry induced by the immersion. Let T_pM denote the tangent space of M at p.

Each T_pM will be considered as a subspace of H. $T_p^{\perp}M$ will denote the orthogonal complement of T_pM.

Let H have inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Then the immersion induces an inner product $\langle \cdot, \cdot \rangle$ and a norm $\|\cdot\|$ on each T_pM. The following notation will be used:

$$S(b, R) = \{X: \|X - b\| \leqslant R\}; \ \|b - M\| = \inf_{p \in M} \|b - p\|;$$

$$S(M, R) = \{X: \|X - M\| \leqslant R\}.$$

Most of the local geometry, like covariant differentiation and curvature of finite dimensional Riemannian spaces generalizes immediately to this setting. In particular, let ∇ and $\tilde{\nabla}$ denote covariant differentiation in H and M, respectively. Then, for each pair of vectors X, Y in T_pM,

$$\nabla_X Y = \tilde{\nabla}_X Y + h(X, Y),$$

where h is a bilinear mapping

$$h: T_pM \times T_pM \to T_p^{\perp}M,$$

the second fundamental form of M at p.

Let c be an arclength parametrized geodesic starting at $p \in M$, with $c(0) = X$. Then

$$\ddot{c}(0) = \tilde{\nabla}_{\dot{c}}\dot{c} = h(X, X) \in T_p^{\perp}M.$$

For $v \in T_p^{\perp}M$ define the symmetric linear operator h_v: $T_pM \to T_pM$ by

$$\langle h_v X, Y \rangle = \langle v, h(X, Y) \rangle; \ X, Y \in T_pM. \tag{2.1}$$

Then the metric curvature of M at p with respect to v (as defined, e.g., in [19 or 1]) corresponds to

$$\sigma_v = \sup_{\|X\| = 1} \langle h_v X, X \rangle$$

and the metric curvature of M at p corresponds to the normal curvature, $\|h\|$ of M at p,

$$\|h\| = \sup_{\substack{\|X\| = 1 \\ X \in T_pM}} \|h(X, X)\|.$$

See [6 or 4] for a discussion of the finite dimensional case. The ∞-dimensional case is similar.

The *folding of M at p*, $\eta(p)$ is defined by

$$\eta(p) = \sup \{\rho \colon S(p, \rho') \cap M \text{ is connected for all } \rho' \leqslant \rho\}.$$

The *folding of M*, $\eta(M)$ is defined by $\eta(M) = \inf_{p \in M} \eta(\rho)$. The concept of folding is introduced to measure how much M "turns back on itself." Note that if M is complete then, as is easy to see, $\eta(M) > 0$ implies that M is imbedded in H and if $\bar{M}$ denotes the closure of M as a subset of H we have $M = \bar{M}$.

Let $b \in H$. The *best approximation operator* $\mathscr{P}_M$ of M is a set-valued mapping which takes b into the set of $p \in M$ such that $\|b - p\| = \|b - M\|$. If $\mathscr{P}_M(b) \neq \varnothing$ then b is said to have a best approximation in M. $\mathscr{P}_M(b)$ contains exactly one element then b is said to have a *unique best approximation* in M.

Let $U(M)$ denote the set of $b \in H$ such that b has a unique best approximation in M and let $DU(M)$ denote the set of $b \in U(M)$ such that $\mathscr{P}_M$ is Frechet-defferentiable at b.

We end this section by stating an auxiliary result which is of interest its own right.

Let h_v be given by (2.1).

LEMMA 2.1. *Let $p = \mathscr{P}_M(b)$ be the unique best approximation in M of some $b \in H$, let $v = b - p \in T_p^{\perp} M$ and assume that $(h_v - I)\colon T_p M \to T_p M$ is invertible. Then $\mathscr{P}_M$ is differentiable in an open neighborhood of b and*

$$D\mathscr{P}_M(b) \cdot \delta b = (I - h_v)^{-1}\, \delta b_{\mathrm{T}},$$

where δb_{T} denotes the tangential part of δb.

Proof. Consider the mapping $\exp_p$,

$$\exp_p \colon T_p M \to M$$

which in the following will be denoted by φ. It is immediate that $\varphi(0) = p$, that

$$D\varphi(x)|_{x=0} \colon T_p M \to T_p M \subset H$$

is the identity mapping of $T_p M$ and that

$$D^2\varphi(x)|_{x=0} = h,$$

the second fundamental form of M at p. Now let $a \in H$ and define $f\colon H \times T_p M \to \mathbb{R}$ by $f(a, x) = \frac{1}{2}\|a - \varphi(x)\|^2$. The derivatives of f at $(b, 0)$ are

$$D_x f\cdot\eta=\langle D\varphi\cdot\eta, v\rangle=0$$

$$D_a D_x f\cdot(\eta,\delta b)=-\langle D\varphi\cdot\eta,\delta b\rangle=-\langle\eta,\delta b_{\mathrm{T}}\rangle$$

$$D_x^2 f\cdot(\eta,\xi)=\langle D\varphi\cdot\eta, D\varphi\cdot\xi\rangle-\langle D^2\varphi(\eta,\xi), v\rangle=\langle (I-h_v)\,\eta,\xi\rangle.$$

Note that $\mathscr{P}_M(b)=(D_x f(b,\cdot))^{-1}(0)$. Thus, at b we have

$$D\mathscr{P}_M\cdot\delta b=(D_x^2 f)^{-1}D_a(D_x f)\cdot\delta b=(I-h_v)^{-1}\,\delta b_{\mathrm{T}},$$

which by assumption is a bounded operator. By the inverse function Theorem [21] applied to $D_x f$, $\mathscr{P}_M$ is defined and differentiable in an open neighbourhood of b. ∎

The proof of Lemma 2.1 is straightforward and has been known for a long time. If λ is such that (λh_v-I) is not boundedly invertible then $p+\lambda v$ is called a focal point of M. See [18, 20, 1–3].

The following corollary is immediate. For a constant α let us write $h_v<\alpha$ if $\langle h_v\xi,\xi\rangle<\alpha$ for all ξ in T_pM; we similarly define $h_v\leqslant\alpha$ and $h_v\geqslant\alpha$.

COROLLARY 2.2. *Let $p\in M$, $v\in T_p^{\perp}M$, $b=p+v$. If $h_v<1$, $\mathscr{P}_{M'},(b)=p$ for some neighbourhood M' of p in M and if $h_v\leqslant 1-\delta$ for some $\delta>0$, then $\mathscr{P}_{M'}$ is differentiable in an open neighbourhood of b.*

On the other hand, if $p\in\mathscr{P}_M(b)$, then with $v=b-p$, $h_v\geqslant 1$.

3. EXISTENCE OF CRITICAL POINTS

The following existence result is simple if the ambient space is finite-dimensional (c.f. [6, Theorem 9]) but here requires a new proof.

THEOREM 3.1. *Assume that M is complete and C^2-immersed in H and that the normal curvature of M satisfies $\|h\|\leqslant 1/\rho<\infty$. Let $\varepsilon>0$. Then for $b\in H$, if $\|b-M\|<\rho$, the distance function $\|b-p\|$: $M\to\mathbb{R}$ has a critical point p^* in M such that $\|b-p^*\|\leqslant\|b-M\|+\varepsilon$.*

Proof. Note that p critical point of $\|b-p\|$ is equivalent to the existence of some $v\in T_p^{\perp}M$ such that $b=p+v$.

For $r>0$, $S^{\perp}M(r)$, the normal sphere bundle of M of radius r is defined as follows. For $(p,v)=T^{\perp}M$, let

$$S^{\perp}M(r)=\bigcup_{p\in M}\{(p,v)\colon v\in T_p^{\perp}M,\ \|v\|\leqslant r\}.$$

Recall that M and $T_p^\perp M$ for $p \in M$ are subsets of H. This enables us to define a mapping $F: S^\perp M(r) \to H$ by $F(p, v) = p + v$. Let $b = p + v$. If $\mathscr{P}_M(b)$ is unique and differentiable then

$$F^{-1}(b) = (\mathscr{P}_M(b), b - \mathscr{P}_M(b)) \qquad \text{and} \qquad DF^{-1} = (D\mathscr{P}_M, I - D\mathscr{P}_M).$$

Choose ε such that $0 < \varepsilon < \rho - \|b - M\|$ and set $r = \|b - M\| + \varepsilon$. Note that then $b \in S(M, r)$ holds. If we set $\Omega = F(S^\perp M(r)) \cap S(M, r)$, then for any $\tilde{b} \in \Omega$, $\tilde{b} = \tilde{p} + \tilde{v}$ for some $\tilde{p} \in M$, $\tilde{v} \in T_p^\perp M$, $\|\tilde{v}\| \leqslant r < \rho$ so $\|h_{\tilde{v}}\| \leqslant r/\rho < 1$. Choose a branch of F^{-1} at $\tilde{b}$ and apply Corollary 2.2 to find that DF^{-1} is a bounded operator and that, by the inverse function theorem [21], F is locally invertible. Thus Ω is open in $S(M, r)$. If we are able to prove that Ω is also closed, then it follows that $\Omega = S(M, r)$ and we are done.

That Ω is closed follows from a standard lifting argument using the continuity and local invertibility of F in Ω (with $\|DF^{-1}\| \leqslant \rho/(\rho - r) < \infty$). See [10, p. 364] or the proof of Lemma 4.3 in [13]. This completes the proof of Theorem 3.1. ∎

COROLLARY 3.2. *Assume that* $\eta > 0$ *and that* $\|h\| < \infty$ *at every point of* M. *Then* $DU(M)$ *contains an open neighbourhood of* M.

Proof. Note that since $\|h\| < \infty$ at each point and $\eta > 0$ we get an open covering $\{O_i\}$ of M in H such that F as defined in the proof of Theorem 3.1 is invertible on each O_i. The open set we were looking for is then $\bigcup O_i$. ∎

4. ON THE SIZE OF $DU(M)$

It is known that if K is a closed subset of H, then $DU(K)$ contains a dense G_δ. Here we show that for imbedded submanifolds this can be improved.

THEOREM 4.1 *Let* M *be a complete* C^2*-imbedded Hilbert submanifold of* H. *Then* $DU(M)$ *contains a dense open subset of* H.

Remarks. (i) Wolfe [22] has proved a similar result for finite dimensional approximatively compact submanifolds. The above result shows that this condition is superfluous. (ii) If the ambient space is finite dimensional, the corresponding result is that the complement of $DU(M)$ has *Lebesgue* measure 0 (c.f. [8]).

Proof. Note that M imbedded and complete means that M is a closed subset of H. It is a result by Asplund [7, p. 45] that $DU(K)$ contains a dense G_δ if K is closed. Thus $DU(M)$ is dense. By applying the inverse function theorem as in the proof of Theorem 3.1, the theorem follows. ∎

5. On the Reach of M

Following Federer [12] we define the reach of M by

$$\text{reach}\,(M) = \inf_{p \in M} \sup \{r \colon S(p, r) \subset U(M)\}.$$

THEOREM 5.1. *Let M be a complete C^∞-imbedded Hilbert submanifold and assume that $\|h\| \leqslant 1/\rho < \infty$. Then*

$$\text{reach}\,(M) \geqslant \min\,\{\rho, \tfrac{1}{2}\eta(M)\} = \mu.$$

Remark. Theorem 5.1 has been proved in the finite dimensional case by Abazoglou [4] with Hilbert space as ambient space and independently by the author [6] with Euclidean space as ambient space.

The proof stated below is a straightforward generalization of that in [6, Theorem 8]. The proof in [4] can also be generalized but requires somewhat more work.

We will need a few lemmas. Throughout, the assumptions of Theorem 5.1 will hold.

LEMMA 5.2. *Let $c\colon \mathbb{R} \to H$ be an arclength parametrized curve and assume that $\|\ddot{c}\| \leqslant 1/\rho$. Assume that $z \in H$ is such that $z \perp \dot{c}(0)$ and $\|z\| = 1$. Then*

$$\|c(o) + \rho z - c(t)\| \geqslant \rho, \quad -\pi\rho \leqslant t \leqslant \pi\rho.$$

For different proofs of this, see [4 and 6]. See also [17, p. 38].

LEMMA 5.3. *Let c be as in Lemma 5.2. Then for $o \leqslant t \leqslant \pi\rho$,*

$$\|c(o) - c(t)\| \geqslant 2\rho \sin\,(t/2\rho).$$

Lemma 5.3 is a restatement of [6, Corollary 3]. For $p, q \in M$ we define the geodesic distance $d(p, q)$ by

$$d(p, q) = \min \int_0^1 \|\dot{c}\|\, dt,$$

where the minimum is taken over all smooth $c\colon[o, 1] \to M$ such that $c(o) = p$ and $c(1) = q$.

In particular, if there exists a geodesic c connecting p and q of length $d(p, q)$ then c is said to be minimizing.

LEMMA 5.4. (Ekeland [11, Theorem B]). *Let $p \in M$.* The set of points connected to p by a minimizing geodesic contains a dense G_δ subset of M.

Remark. Lemma 5.4 is the only place where the assumption that M is of class C^∞ is used and it is to be expected that this, too, holds for C^2-manifolds.

LEMMA 5.5. *Let $p \in M$ and $z \in T_p^\perp M$, $\|z\| = 1$. If $b' = p + r'z$, $|r'| < \rho$ then p is the unique best approximation to b' in $\Omega = \{p' \in M\colon d(p, p') \leqslant \pi\rho\}$.*

Proof. Let $p' \in \Omega$ be connected to p by a minimizing geodesic as in Lemma 5.3. Note that if c is an arclength parametrized geodesic of M then

$$\|\ddot{c}\| = \|h(\dot{c}, \dot{c})\| \leqslant \|h\| \leqslant 1/\rho. \tag{5.1}$$

Lemma 5.2 now implies that $\|b' - p'\| > r'$. Using Lemma 5.4 we find that points with this property are dense in Ω. This completes the proof of Lemma 5.5. ▌

Proof of Theorem 5.1. Choose a $b \in S(M, \mu - \delta)$ for some $\delta < 0$ and set $r = \|b - M\| \leqslant \mu - \delta < \rho$. By Theorem 3.1, for any $\varepsilon > 0$ there exists a $p \in M$ which is a critical point of $\|b - p\|$ and such that $\|b - p\| \leqslant \|b - M\| + \varepsilon$. Choose ε such that $\varepsilon < \delta$ and $2r + \varepsilon < \pi r$.

We will derive a contradiction from the assumption that p is not the unique best approximation to b in M, i.e., that there exists a $q \in M$, $q \neq p$ such that $\|b - q\| = r$.

First assume that $\mu = \rho$, i.e., $\eta(M) \geqslant 2\rho$. By assumption,

$$\|p - q\| \leqslant \|p - b\| + \|q - b\| \leqslant 2r + \varepsilon < 2\rho \leqslant \eta(M).$$

so by the definition of $\eta(M)$ there is a curve γ connecting p and q which lies entirely in $S(p, 2r + \varepsilon) \cap M$.

By Lemma 5.5, $d(p, q) > \pi r$ must hold. Using (5.1), Lemma 5.3, and Lemma 5.5, one sees that γ must contain points not in $S(p, 2r + \varepsilon)$, a contradiction since $\varepsilon > 0$ was arbitrary.

On the other hand, assume that $\mu = \eta(M)/2$, i.e., $\rho \geqslant \eta(M)/2$. It is sufficient to consider $M \cap S(p, \eta(M))$, since $\|x - p\| > \eta(M)$ implies $\|x - b\| > \mu > r + \varepsilon$ for x in M.

Let $p' \in M$ be connected to p by a minimizing geodesic c of length $s \leqslant \pi\rho$. By (5.1) it follows that $\|\ddot{c}\| \leqslant 1/\rho$ and now Lemma 5.3 implies

$$\|p - p'\| \geqslant 2\rho \sin(s/2\rho). \tag{5.2}$$

By the definition of $\eta(M)$, for any $\alpha > 0$, $M \cap S(p, \eta(M) - \alpha)$ is connected. Apply Lemma 5.4 and relation (5.2) to find that $p' \in M \cap S(p, \eta(M) - \alpha)$ implies that $d(p, p') \leqslant \pi\rho$. But Lemma 5.5 implies that p is the unique best approximation to b in $\{p' \in M\colon d(p, p') \leqslant \pi\rho\}$. This yields a contradiction to the assumption that p is not the unique best approximation to b in M since $\alpha > 0$ was arbitrary and by assumption, $r < \eta(M)/2$. But $\delta > 0$ was arbitrary. This completes the proof of Theorem 5.1. ∎

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